Stability of boundary layer flows in different regimes

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ABSTRACT

The complete understanding of flow stability has a deep impact on the capability of predicting and controlling transition. On the other hand, the harsh hypersonic environment requires a deep understanding of the different phenomena occurring simultaneously around the vehicle. Most often these phenomena interact with each other making accurate prediction of the flow field challenging. In order to better appreciate these different aspects this review present the specific features of boundary layer stability at different regimes

Contents

1.0 INTRODUCTION

After the influential paper of Reynolds [1] the evidence of the turbulent behavior of flows under special circumstances gave momentum to the study of turbulent phenomena and *transition* from laminar to turbulent flow. It was evident in the original paper that the appearance of laminar or turbulent condition was linked to a threshold velocity. A common idea at the time linked this change to small disturbances acting on the mean flow. For this reason the main driving conjecture behind the physics of the phenomenon was whether these disturbances would be damped or get amplified (see for instance Rayleigh [2]). In this sense it is possible to call a flow respectively *stable* or *unstable*. Reynolds [3] proposed the idea that a flow, initially laminar, would become turbulent once the amplitude of the disturbances exceeds a certain threshold. An interesting historical review of the original contributions of Reynolds has been published by Jackson and Launder [4] and a similar summary on the origin of the Reynolds number is reported by Rott [5].

For a boundary layer on a flat plate, the first calculations of the critical Reynolds number were obtained only several decades later: first by Tollmien [6], who retrieved a value of $Re = 420$ and by Schlichting [7] who found the value $Re = 575$. In the '70s Jordinson numerically determined the value of 520, while at the present date 519 is the commonly accepted value.

A few years before Tollmien, prior to moving definitely to other promising areas of physics, Heisenberg [8] wrote a paper about the Poiseuille flow, as an outcome of his PhD thesis under Sommerfeld's supervision. He gave an approximate solution to the viscous problem stating that viscosity can have a destabilizing effect and gave a qualitative picture of the stability curve. This counterintuitive result was not completely accepted by the scientific community and led to a rather heated debate in the following years until Krylov [9] confirmed the pioneering result of Heisenberg.

A key idea to stress is that unstable flows allow laminar solutions but any perturbation, even arbitrarily small, will get amplified, leading eventually to chaotic behavior and turbulence. To better understand this behavior, one can imagine a ball on top of a convex surface (top half of a sphere) or inside a concave surface (bottom half of a sphere): the ball can stay in its configuration on the convex surface only if there are no perturbations, in any other case it will roll indefinitely far from the original position.

As the early investigators guessed, transitional flows could be imagined as the sum of laminar mean flow and perturbations. Following the usual approach used in the study of stability of physical systems, one assumes the disturbances to be small (in comparison to the mean flow) and linearizes the problem around the base solution, i.e. the laminar mean flow.

The stability of a boundary layer flow is usually affected by many parameters, such as, for instance, geometry, pressure gradient, temperature, free stream noise and receptivity. Each of these parameters has a specific effect on the stability of a flow and according to their combination one could observe different paths/types of transition. In Fig. 1, an illustration is given of these different paths for increasing free stream disturbance level. Indeed, what is usually considered a single phenomenon is a set of intertwined physical effects that could bring the flow from an unstable laminar configuration to a more stable turbulent one. Among these many possibilities this chapter will refer mainly to modal growth.

On the other hand, stability properties of a flow depend on its regime, therefore a subsonic boundary layer is destabilized in a different way than the one of a hypersonic flow. These differences will be highlighted here mostly by means of a linear stability analysis, even if more complex investigations could be performed on similar geometries. Also, for the sake of consistency, a simple flat plate boundary layer flow with no pressure gradient is considered, in order to clear any possible ambiguity rising from the use of a different geometry. It is worth noting that most of the applications in high speed flow involve mainly cones (with sharp or blunt nose) and to a lesser extent flat plates. Nevertheless the stability of supersonic flows on cones can be studied with the same linear stability techniques; in case of cones that are not aligned with the free stream velocity some precautions on the transition front prediction should be taken (see Perraud *et al.* [11]).

The choice of this method implies that only the primary source of stability could be determined and studied. Secondary instabilities would require different theoretical tools not considered in the present lecture.

2.0 INCOMPRESSIBLE BOUNDARY LAYER STABILITY

The work of Rayleigh [12] has provided, first of all, an equation to understand some elementary properties of flow stability. The main limitation of his approach had its origin in the, at the time, common misconception that viscosity could not be a source of instability in the flow and therefore could be neglected in the mathematical

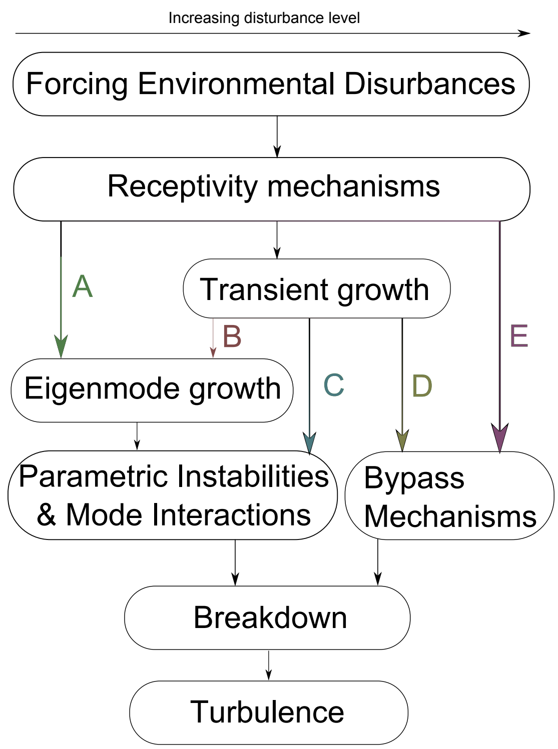


Figure 1: Paths to transition for wall bounded flows (Reshotko [10])

model. From the Rayleigh equation one can extract the so-called Rayleigh's inflection point criterion that states: if there exists a phase speed with $c_i > 0$ then the second spatial derivative of the velocity, $U''(y)$, goes to zero at one point in the domain. Despite this inspired early work, it is necessary to skip to the dawn of the 20$^{\text{th}}$ century to have a formalized version of the Navier-Stokes equations linearized around a mean flow. This equation for an incompressible boundary layer has been derived independently by Orr [13, 14] and Sommerfeld [15] since then carrying their name:

$$\frac{\mathrm{d}^4\tilde{v}}{\mathrm{d}y^4} - 2k^2\frac{\mathrm{d}^2\tilde{v}}{\mathrm{d}y^2} + k^4\tilde{v} - iR\left[\left(\alpha\overline{U} + \beta\overline{W} - \omega\right)\left(\frac{\mathrm{d}^2\tilde{v}}{\mathrm{d}y^2} - k^2\tilde{v}\right) - \left(\alpha\frac{\mathrm{d}^2\overline{U}}{\mathrm{d}y^2} + \beta\frac{\mathrm{d}^2\overline{W}}{\mathrm{d}y^2}\right)\tilde{v}\right] = 0\,. \tag{1}$$

Being a fourth order equation, one must prescribe four boundary conditions. It is usually considered that the velocity and its wall-normal derivative are zero at the wall, because of the no penetration condition for the normal velocity $\tilde{v}$ and the no-slip condition for the streamwise velocity perturbation. For unbounded flows or semi-bounded flows, velocity perturbations could be considered negligible in the potential core of the flow, far away from the boundary and therefore there is always a set of homogeneous boundary conditions. Di Prima and Habetler [16] were the first to prove that for bounded flows the spectrum of the Orr-Sommerfeld equation is made of an infinite number of discrete eigenvalues that forms a basis in the state space, meaning the spectrum is complete (Grosch and Salwen [17], [18] and Craik [19]). This is true only for viscous perturbations whereas for inviscid flows only a continuous spectrum is possible. In case of unbounded or semi-bounded flows, arbitrary initial disturbances cannot be written in terms of a finite number of discrete modes; therefore the continuous spectrum completes the function space.
Prandtl [20] was the first to realize the existence of two flow regions around a smooth body: an outer part

following Euler's potential theory and a boundary layer where the viscosity is the main factor governing the flow. The development of this concept inspired his Ph.D. student Blasius [21] to develop a model for the incompressible boundary layer. The equation for a flat plate without any pressure gradient (that is: constant velocity U_e in the potential flow) reads:

$$f''' + ff'' = 0, \tag{2}$$

where $f = f(\eta)$ and $\eta = y\sqrt{Re/x}$. The velocity is represented by the function $U = U_e f'(\eta)$ and the boundary conditions are

$$f(0) = f'(0) = 0, \quad f'(+\infty) = 1. \tag{3}$$

No analytical solution is known for this equation and hence a numerical integration is necessary. The first solution, provided by Blasius himself, used a power series allowing the estimation of the force on the plate. A more standard technique consists in a shooting method, that transforms the boundary value problem into an initial value problem with an artificial boundary condition $f''(0) = s$. In a second step, a Newton iteration is repeated until the condition $f'(+\infty) = 1$ is also respected. The main intention of Blasius was to solve the turbulence problem and despite his disappointment (see Hager [22]), his research paved the way for the developments in this field.

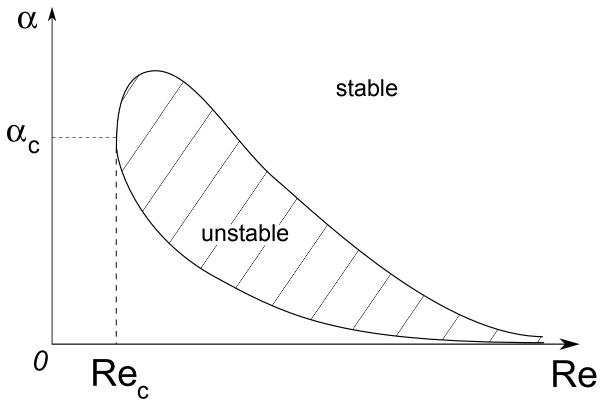
In boundary layer flows without pressure gradients, as apparent from the solution of eq. (2), the profile does not show an inflection profile and therefore Rayleigh's theorem shows there is no inviscid instability, and the Orr-Sommerfeld equation has to be solved directly.

It is worth noting that, strictly speaking, the Orr-Sommerfeld equation is valid only for a purely parallel and steady incompressible flow. The steady (mean) flow should also be considered as an exact solution of the Navier-Stokes equations. This would tremendously restrict the field of application of the theory only to Couette and Poiseuille flow. Nevertheless, it was Tollmien [6] who applied it to boundary layers under the assumptions that the streamwise variations of the streamwise velocity component U are negligible, being of the order[1] $\mathcal{O}(Re_x^{-1/2}) = \mathcal{O}(Re_\delta^{-1})$.
This approach is valid as long as the Reynolds number is sufficiently high, which in turn implies that the considered profile has to be far form the leading edge, where the quasi-parallel flow assumption is likely to fail. Furthermore the wavelength of the most unstable disturbance should be small enough, so to be influenced negligibly by the growth of the mean flow profile. The results obtained by Tollmien have been criticized for several decades, at least until the experimental validation from Schubauer and Skramstad [23][2]. The issues related to non-parallelism are numerous and not entirely understood; during the past decades, several researchers contributed to an estimation of their effect. This notably originated the work on Parabolized Stabilty Equation (Bertolotti [24, 25], Herbert[26]) and Biglobal stability analysis (Theofilis [27]).

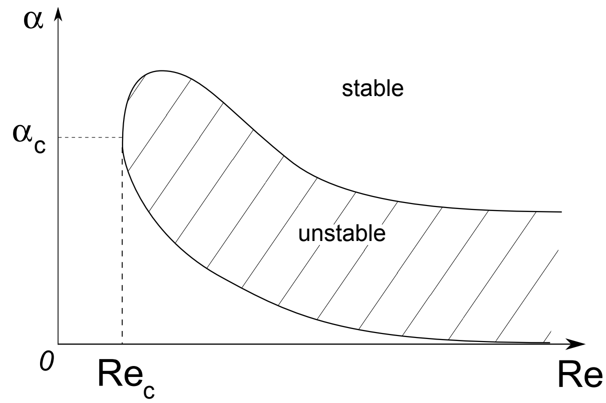
In hindsight, the role of viscosity is two-fold, as it acts both as a stabilization factor, due to the energy dissipation, as well as a destabilizing one. Viscosity, which plays an important role in the boundary layer, leads the transfer of energy from the mean flow to the critical layer through Reynolds stresses. A simplification of this observation is that if a flow is unstable for an inviscid fluid, the most unstable eigenvalue of the viscous case should tend to the inviscid one as the Reynolds number approaches infinity. On the other hand, if a flow is stable at the inviscid limit, then its least stable eigenvalue may be different from the value of the viscous case for $Re \to \infty$. If the Reynolds number is sufficiently small, the flow is usually stable (with the exception of an unbounded shear layer). This trend is summarized in Fig. 2 where typical neutral stability curves are displayed:

[1] $Re_\delta = U\delta/\nu = (Ux/\nu)^{1/2} = Re_x^{1/2}$ where δ is a boundary layer thickness expressed, for instance in term of Blasius length $\delta = (\nu x/U)^{1/2}$.

[2] Part of the paper was previously published in April 1943 as NACA Advance Confidential Report.



(a) Qualitative neutral stability curve for a flow stable at the inviscid limit



(b) Qualitative neutral stability curve for a flow unstable at the inviscid limit

Figure 2: typical neutral curve for a flow stable at the inviscid limit

it is closing for stable velocity profiles in the inviscid limit and it is open if the flow is unstable at the inviscid limit.

Starting from the three-dimensional Orr-Sommerfeld eq.(1) it is easy to note that the two-dimensional counterpart, obtained by setting $\beta = 0$, has the same solution provided that the corresponding two-dimensional wave number is

$$\alpha_{2D} = \sqrt{\alpha^2 + \beta^2}\,, \tag{4}$$

$$\alpha_{2D} Re_{2D} = \alpha Re\,, \tag{5}$$

from which the following relation could be obtained

$$Re_{2D} = Re \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}\,. \tag{6}$$

It follows that if a three-dimensional flow has an unstable mode at a certain Reynolds number, a corresponding two-dimensional mode is unstable at an even lower Reynolds number. This statement is usually known as Squire's theorem. An incompressible boundary layer will start developing two-dimensional disturbances first, called Tollmien-Schlichting (TS) waves. Only at later stage they will turn into three-dimensional disturbances. Numerical experiment from Orszag and Patera [28] found that Tollmien-Schlichting waves superimposed on a boundary layer (representing a boundary layer where the primary instability already evolved) form an unstable flow and that such instability is inherently three-dimensional. This finding should not be confused with Squire's theorem as that theorem refers to primary (TS) instabilities, not secondary ones.

3.0 COMPRESSIBLE BOUNDARY LAYER STABILITY

In 1947, the Bell X1, piloted by C. Yeager, was the first aircraft to break the sound barrier in a level flight. Many other pilots claimed to have reached Mach 1 in special conditions during the second world war and it is also worth a note that, when fully operational during the last years of World War II, the German V-2 was routinely reaching Mach 4 during its descent. The rising interest in supersonic vehicles and all the problems related to supersonic flight encouraged the scientific community to extend the study of stability to compressible flows.

The development of compressible linear stability equations follows the usual procedure that is applied to the incompressible counterpart. The main formal difference lies in the inability of combining the five equations (continuity, three momentum and energy equations) into a single one. Therefore the system is more complex and difficult to solve.
The original findings from the Orr-Sommerfeld equation do not always apply. Unlike the incompressible linear stability theory (LST), energy transfer is more relevant and thermodynamic, as well as transport properties become more important. Therefore physical properties such as viscosity, μ, bulk viscosity, λ or thermal conductivity, k are not constant anymore and they have to be perturbed

$$\mu = \overline{\mu} + \mu' , \tag{7}$$

$$\lambda = \overline{\lambda} + \lambda' , \tag{8}$$

$$k = \overline{k} + k' . \tag{9}$$

Quantities are made dimensionless usually with respect to the Blasius length scale $l_{ref} = (\nu x/U_e)^{1/2}$, velocity U_e, density ρ_e, pressure by $\rho_e U_e^2$, time by l/U_e and temperature by T_e. For viscosity (both first and second coefficient) the value corresponding to the temperature T_e is taken as reference.
Even though several sources can be found displaying the resulting set of equations, they are reported in the appendix A for the reader's convenience.
Despite their complicated form these equations can be cast in a simple standard form

$$\left(AD^2 + BD + C\right)\chi = 0, \tag{10}$$

where D represents the derivative with respect to the wall normal direction and the unknown variables vector reads

$$\chi = [\tilde{u}, \tilde{v}, \tilde{p}, \tilde{T}, \tilde{w}]^T . \tag{11}$$

The matrices A, B and C contain the wavenumber α or the frequency ω. It is clear that, similar to the incompressible case this is an eigenvalue problem and, once discretized, all the standard solution techniques can be used. A compressible boundary layer on a flat plate, with or without a pressure gradient, can be described by only two equations of the following form (see Cebeci and Smith [29])

$$\left(c_1 f''\right)' + ff'' + \beta_1(c_2 - f'^2) = 2\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) , \tag{12}$$

$$\left(a_1 g' + a_2 f' f''\right)' + fg' = 2\xi\left(f'\frac{\partial g}{\partial \xi} - g'\frac{\partial f}{\partial \xi}\right) , \tag{13}$$

where

$$f' = \frac{u}{U_e}; \quad c_1 = \frac{\rho\mu}{\rho_e\mu_e}; \quad c_2 = \frac{\rho_e}{\rho};$$
$$g = H/H_e; \quad a_1 = c_1/Pr; \quad \beta_1 = (2\xi/u_e)\left(du_e/d\xi\right);$$
$$a_2 = \frac{(\gamma-1)M^2}{1+\left(\frac{\gamma-1}{2}\right)M^2}\left(1 - \frac{1}{Pr}\right)c_1 ,$$

where ξ is the transformed streamwise curvilinear coordinate[3]. Eq.(12) represents the momentum equations and eq. (13) comes from the energy balance. If the small dependency of the flow on the streamwise coordinate is neglected, therefore neglecting the right hand side of eq.(12), a self similar solution is possible.

The first theoretical investigation on the subject was performed by Lees and Lin [30] for a two-dimensional perfect gas temporal problem. They extended Rayleigh's theorem to compressible flows and they discussed, first, the supersonic waves moving in the free stream by means of an energy method. They also discovered the role of the quantity

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\rho\frac{\mathrm{d}U}{\mathrm{d}y}\right) = 0 \; , \tag{19}$$

which is the compressible counterpart of the $\mathrm{d}^2U/\,\mathrm{d}y^2 = 0$ criterion for the incompressible flows.
The most important consequence of eq.(19) is that a compressible flat plate boundary layer, unlike its incompressible counterpart, may be unstable to inviscid disturbances. This fact modifies the neutral stability curve qualitatively from fig. 2(a) to fig. 2(b). Despite this interesting result the community had to wait for the extensive work of Mack [31–36] before the stability in compressible flows could be completely characterized.
The role of the Mach number is of primary importance and a convenient quantity in the theoretical treatment is the definition of the relative Mach number

$$\hat{M} = \frac{(\alpha U + \beta W - \omega)M}{\sqrt{T}\sqrt{\alpha^2+\beta^2}} \tag{20}$$

that represents the local Mach number of the mean flow along the direction of the wavenumber vector relative to the phase velocity (see the review of Mack [37]). This allows the definition of a disturbance traveling inside the boundary layer as subsonic, sonic or supersonic if the relative Mach number is respectively smaller than, equal to, or greater than one. This definition applies at the boundary layer edge, therefore it also reads

- subsonic: $U - c_r < a$
- sonic: $U - c_r = a$
- supersonic: $U - c_r > a$.

Similarly to what was already proven for incompressible conditions, Lees and Lin [30] could demonstrate that the presence of a generalized inflection point as in eq. (19) is a necessary condition for the existence of an unstable wave for a subsonic disturbance, necessary and sufficient for the existence of neutral subsonic waves and it is sufficient for the unstable wave to have a unique wave number. The treatment implied that the critical layer, where the wave speed equals the flow speed, is close to the wall so that the respective velocity was small (i.e. subsonic). This assumption implied that density or temperature had a small rate of change allowing for an

[3]Two subsequent transformations are applied to the Navier-Stokes equations. The first is the Mangler transformation

$$d\overline{x} = (r_0(x)/L_r)^{2k}\, dx \tag{15}$$
$$d\overline{y} = (r(x,y)/L_r)^{k}\, dy \tag{16}$$

where L_r is a reference length, r_0 is the radius of the axisymmetric surface. Next the Levy-Lees transformation is applied, yielding

$$d\xi = \rho_e\mu_e U_e d\overline{x} \tag{17}$$
$$d\eta = \left[\rho U_e/(2\xi)^{1/2}\right] d\overline{y} \tag{18}$$

where ρ_e, μ_e are respectively the density and the viscosity at the boundary layer edge. More details can be found in Cebeci and Smith [29]

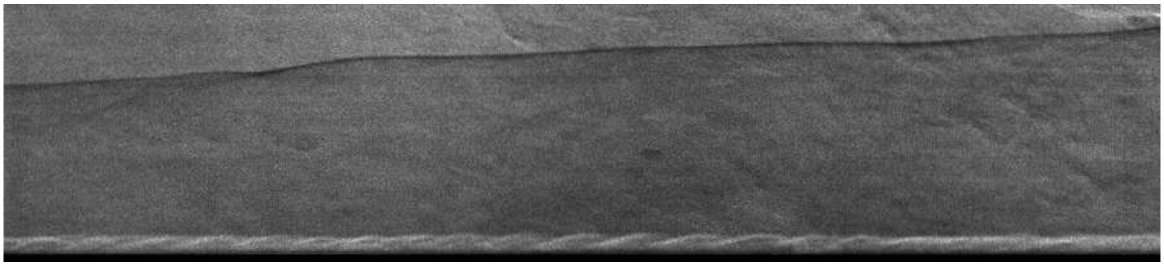


Figure 3: Schlieren picture of 2nd mode from Grossir [40]

"incompressible" treatment of the disturbances. Already in the work of Dunn and Lin [38] it was argued that this is not applicable to supersonic flows with high Mach numbers. Their argument was based on the simple consideration that at high Mach numbers, temperature effects would be more relevant.

For an insulated flat plate, considering an approximate $Pr = 1$ for $\gamma = 1.4$, the relative Mach number is greater than, equal to or smaller than unity if the external Mach number is respectively bigger than, equal to or less than 2.2 (see Lees and Reshotko [39]). Under the aforementioned conditions, for $M_e \geq 2.2$ there exists an area of relative supersonic flow close to the wall surface. This makes inviscid perturbations possible and relevant.
Another interesting result showed that damped waves, either subsonic or supersonic, evolve away from the boundary layer into the freestream while the amplified counterparts are directed towards the boundary layer. In case of neutral waves, they propagate along the streamwise direction or toward the boundary layer if they are respectively subsonic or supersonic.

Mack [32] was the first to find the multiple solutions where the absolute value of the relative Mach eq. (20) is greater than one. Writing Rayleigh's equation in terms of the pressure perturbation one has

$$\tilde{p}'' - \ln\left(\hat{M}^2\right)' \tilde{p}' - \left(\alpha^2 + \beta^2\right)\left(1 - \hat{M}^2\right)\tilde{p} = 0 \; . \tag{21}$$

It is easily understood from eq. (21) that for a supersonic relative Mach number, when neglecting the second term, an inviscid disturbance behaves as a wave equation, carrying multiple solutions. The first mode that belongs to this family is usually the least stable/most unstable and it is traditionally called Mack's second mode (see the characteristic rope-like shape in Fig. 3).

In light of this, linear stability of compressible flow is characterized by Mack's first mode (usually considered as a compressible extension of the Tollmien-Schlichting incompressible waves) and Mack's second mode. Mack was also the first to discover that for compressible flows, the first mode reaches its maximum growth for a non-zero wave angle.
It is worth noting that Mack's first and second modes are currently obsolete nomenclature originated when direct simulation was not accessible and receptivity phenomena were poorly understood mechanisms. As a matter of fact, Mack's mode definition is not consistent with standard mathematical terminology. The use of the term *fast* modes (F_+ and F_-) and *slow* mode is more rigorous even if the traditional nomenclature could be kept alive in the interpretation of experimental and linear stability data. An analogue within these two different nomenclatures is out of the scope of the present lecture and its treatment is reviewed in the paper of Fedorov and Tumin [41] and references therein.

Another family of modes is also present in a range of phase speed $1 \leq c \leq 1 + 1/M$ that has an infinite number of corresponding wave numbers. This family does not show any inflection point and is therefore called non-inflectional. The waves at $c = 1$ are always followed by unstable waves at $c < 1$ and for this reason a supersonic flow, with respect to the relative Mach number, can be considered always unstable to inviscid perturbations, independently of any other parameter. This feature does not have any counterpart in an incompressible flow.

Compressible flows are strongly influenced by temperature and display different behaviors according to their specific condition. During his work, Mack [37], identified several families of instabilities which are all behaving differently to temperature and heat flux. It is a well established result that cooling has a stabilizing effect for the first mode. This was first deduced on the basis that a compressible boundary layer with an adiabatic wall has only one inflection point, while another generalized inflection point appears as the surface temperature is reduced. This second inflection point is slowly rising until it reaches the same height as the first canceling each other out. As a consequence there is no unstable first mode disturbance. This same reasoning does not apply to Mack's second mode, whose existence is controlled solely by the appearance of a supersonic relative Mach number. In this case it has been shown numerically that cooling the wall is increasing Mack's second mode growth rate. It is also worth noting that stability properties of these flows depend also on the free stream temperature, and different values can lead to substantially different neutral stability curves (see Özgen and Kırcalı [42]).

Analysis of the continuous spectrum of a compressible boundary layer was performed by Tumin and Federov [43] who found that it was composed of seven different branch cuts in the complex wavenumber plane, four of which contribute to the flow field (two acoustic branches, one entropy and one vorticity branch). The other three have a very large imaginary part and decay over a small distance. Later, Balakumar and Malik [44] found that the field near the source of disturbance is represented by the continuous spectrum and the first few unstable modes. They conjectured that similarly to the incompressible case, a compressible boundary layer has only a finite number of discrete modes, nevertheless they could not prove it.

4.0 HYPERSONIC BOUNDARY LAYER STABILITY

Unlike the differences between supersonic and subsonic flows where the physics and the nature of the equations change substantially, the transition from high supersonic speed to low hypersonic and beyond happens without any sudden change. Nevertheless a few new interesting features appear in a hypersonic flow: a strong shock in front of the body, the entropy layer, high heating at the wall, strong viscous interaction and eventually chemical reactions (see Anderson [45]). Each of these aspects has a specific influence on the flow stability, though the most notable effect is due to the increase in temperature. A great amount of energy is transformed into heat and if temperature passes a certain threshold, air cannot be considered anymore as a single gas but rather as a gas mixture. Above $2000K$ molecular oxygen starts dissociating and at $4000K$ the same happens to molecular nitrogen. At even higher temperature ionization occurs as well, increasing the number of species to be accounted for. Depending on the chemical activity ongoing within the flow around the hypersonic vehicle (and also on its altitude) different regimes can be identified, ranging from chemical equilibrium to Thermo-Chemical Non-Equilibrium (TCNEQ).

The choice of one model over the other depends on the flow conditions. Excited molecules interact among each other by means of collisions and both the exchange of energy and the chemical reactions happen because of them. Thermodynamic conditions affect this process quite intuitively; high pressure, for instance, packs more molecules and atoms in the same volume, causing more collisions to happen; on the same line, high temperature increases the level of excitation of the molecules resulting in a higher number of collisions as well. The direct consequence is that more equations should be taken into account; a mass equation should be added for each considered species, and, in case of a thermo-chemical non-equilibrium flow, more energy balance equations should be taken into account. On the other hand, the highest speeds are achieved in the highest and

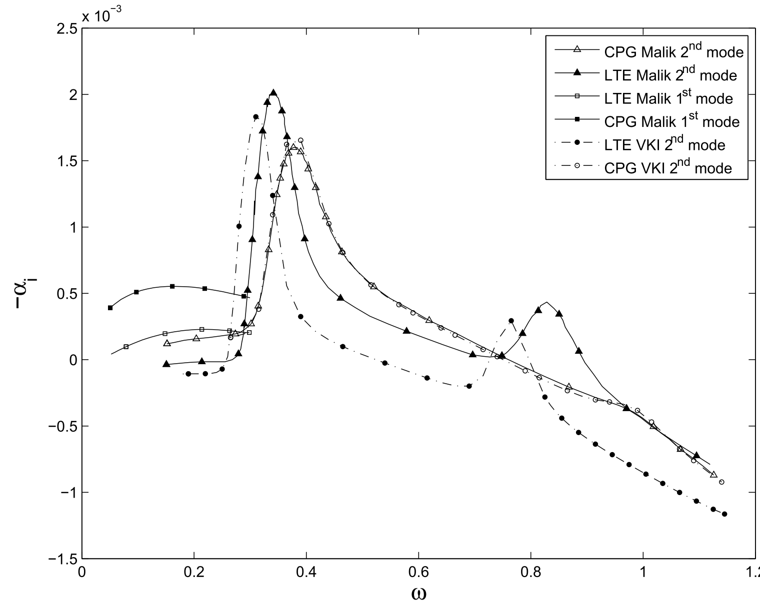


Figure 4: comparison of growth rates for 1st and 2nd modes for an adiabatic Mach 10 flow. Test case from [46]

most rarefied layers of the atmosphere, where, given the small density, Reynolds numbers are low and therefore transition is less likely to happen.

Malik and Anderson [46] have been among the first to analyze stability of boundary layers in hypersonic flows. They assumed Local Thermodynamic Equilibrium (LTE), which allowed the use of a self similar solution of the boundary layer. Comparison against a calorically perfect gas for a similar case on an adiabatic wall showed a much lower temperature at the wall and a thinner boundary layer, probably due to an effective smaller specific heat ratio, γ and a smaller viscosity. Stability of the second mode is altered moving its frequency peak to slightly lower frequencies with higher growth rates. On the contrary, for Mack's first mode disturbances, equilibrium chemistry shifts the peak to slightly higher frequencies with lower growth rates (see Fig. 4).

The unique features of hypersonic flows, especially the strong bow shock, renders the use of self-similar boundary layer profiles nothing more than an academic test case. As a matter of fact, even the work of Malik and Anderson [46], where the correct transport properties are considered, is not an accurate representation of a real flow. Different approaches have been proposed over the years to take into account the gradient normal to the wall and higher order boundary layer theories have been proposed (notably by Van Dyke [47]). Nevertheless, the inviscid vortical flow (entropy layer) prevents a correct matching between the viscous solution and the inviscid one. A more recent approach was developed by Aupoix *et al.* [48] where, along Van Dyke's original intuition, the inner and outer solution are hierarchized. The starting point is a first order solution of an Euler flow. Then the first order boundary layer is computed, followed by an updated higher order Euler calculation, which takes into account the boundary layer displacement and finally a second order defect boundary layer equations.

Stuckert and Reed [49] used the Parabolized Navier-Stokes (PNS) equations with a shock-fitting scheme in order to cancel unwanted oscillations. A comparison of equilibrium and non-equilibrium reacting flows against a standard perfect gas calculation showed that first modes are adversely affected by the equilibrium assumption revealing a stabler behavior with respect to non-equilibrium air and perfect gas. For second modes, equilibrium and non-equilibrium mixtures showed comparable growth rates with a lower frequency peak, similar to what

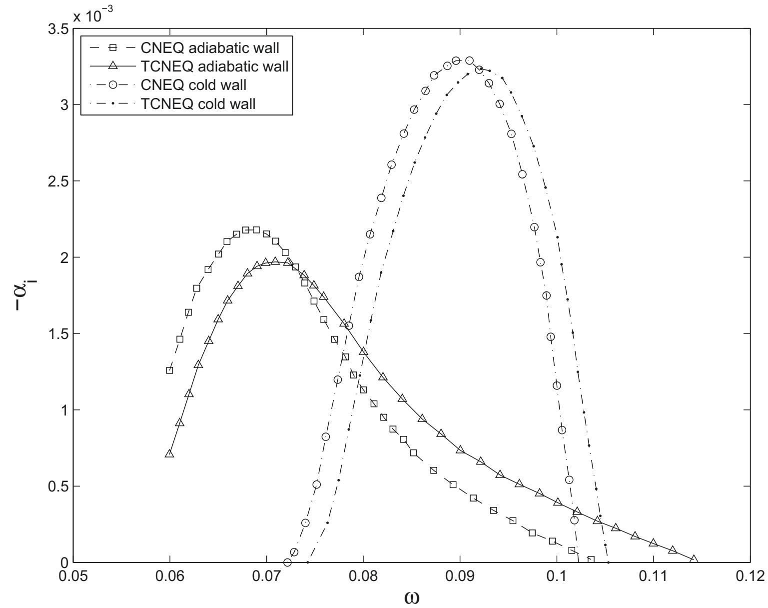


Figure 5: comparison of 2nd modes growth rates between adiabatic and cold Mach 10 flow. Redrawn from [51]

was found by Malik and Anderson [46]. This is due to a reduction of the temperature caused by the chemical reactions (and the consequent increase in specific heat constants).

It is important to realize that real vehicles have very often conical shapes. For this reason, hypersonic ground testing is mainly focused on cones. As transition and stability analysis on cones alone is far beyond the objective of the present lecture, only the work of Johnson *et al.*[50] is mentioned. They could reproduce the Reynolds number increase with increasing freestream total enthalpy. As the rate of increase in transition Reynolds number is greater in air than in nitrogen alone, it is inferred that the lower dissociation energy of molecular oxygen damps the energy fluctuations, thus lowering the corresponding growth rates. Thanks to the more complex thermodynamic model, including thermo-chemical non equilibrium, it was possible to test different chemistry effects showing that according to the type of reactions (endothermic or exothermic) the resulting flow could be stabilized or destabilized.

A similar work from Hudson *et al.* [51] compared thermo-chemical non-equilibrium (TCNEQ), chemical non-equilibrium and chemical equilibrium linear stability solvers. An adiabatic computation at Mach 10 showed that the TCNEQ air model is responsible for the least unstable disturbances due to the stabilizing effect of the higher wall temperature (for Mack's second mode disturbances). The most unstable (oblique) first mode was found to be most destabilized by the TCNEQ model and least destabilized by chemical equilibrium. On the other hand the second mode showed a different behavior: the chemical non-equilibrium model was slightly destabilizing, whereas TCNEQ was stabilizing for the specific conditions considered. Moreover, it was found that the wall cooling was more important than any chemical model used in the calculations (see Fig. 5). The theoretical analysis confirmed that even for hypersonic chemically reacting flows a cooled wall induces a decrease in the transition Reynolds number.

As already mentioned, the strong shock appearing in front of the vehicle represents a well known feature of hypersonic flows. Its effect on boundary layer stability has been studied by several authors; probably Petrov [52] was the first to use a linearized Rankine-Hugoniot relation as the shock boundary condition. Cowley

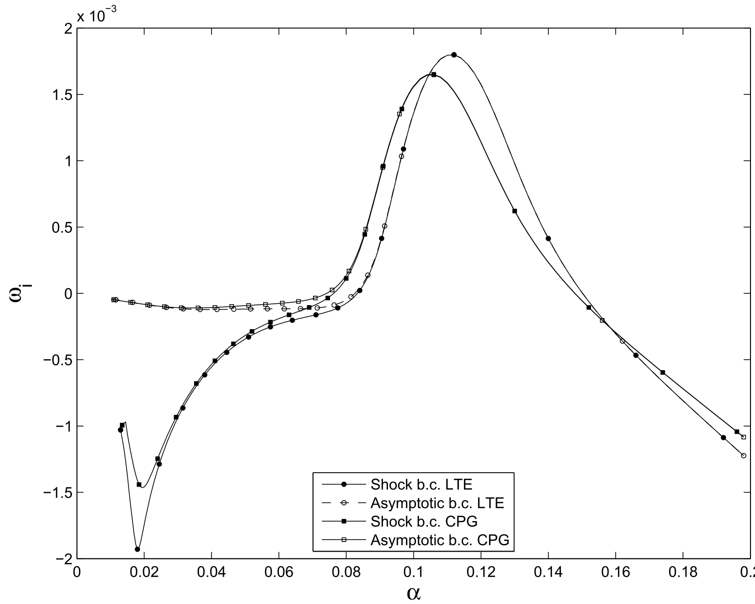


Figure 6: effect of shock on stability of a Mach 8 flow under LTE assumptions

and Hall [53] used the triple deck theory together with a Rankine-Hugoniot boundary condition. Chang *et al.* [54] and Herbert and Esfahanian [55] used the unsteady jump condition to derive the boundary condition that models the shock influence. Stuckert [56] formulated a slightly different approach, although starting from unsteady Rankine-Hugoniot as well. Pinna and Rambaud [57] applied a similar technique to flows using the local thermodynamic equilibrium assumption. In many cases, approximated self-similar boundary layer profiles were used and cropped at the shock location predicted by a simple inviscid calculations.

In case the shock is sufficiently close to the boundary layer, perturbations do not vanish in the far field but they interact with the shock itself. For this reason, the standard homogeneous boundary conditions at infinity cannot be applied anymore. A set of boundary conditions could be implemented, representing the role of the shock. This set of equations comes from a linearization of the unsteady Rankine-Hugoniot jump relations that could be generally written as:

$$\frac{\partial f}{\partial t}[\mathbf{Q}] + \frac{\partial f}{\partial x}[\mathbf{E}] - [\mathbf{F}] + \frac{\partial f}{\partial z}[\mathbf{G}] = 0 \; ; \tag{22}$$

where

$$\mathbf{Q} = [\rho, \rho u, \rho v, \rho w, e]^{tr} \tag{23}$$

$$\mathbf{E} = [\rho u, \rho u^2, \rho uv, \rho uw, (e+p)u]^{tr} \tag{24}$$

$$\mathbf{F} = [\rho v, \rho uv, \rho v^2, \rho vw, (e+p)v]^{tr} \tag{25}$$

$$\mathbf{G} = [\rho w, \rho uw, \rho vw, \rho w^2, (e+p)w]^{tr} \tag{26}$$

and $a = \mathrm{d}f/\,\mathrm{d}x$ is the shock slope.

The Rankine-Hugoniot equations are linearized with respect to the mean shock position $\overline{y}_s = f(x, y, z)$ and by introducing harmonic waves one gets:

$$i(\alpha[\overline{E}] + \beta[\overline{G}] - \omega[\overline{Q}]) + \alpha[\tilde{E}] - [\tilde{F}] = 0 \tag{27}$$

where $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{F}}$ are

$$\tilde{\mathbf{E}} = \begin{bmatrix} \rho_2\tilde{u} + \tilde{\rho}u_2 \\ \tilde{\rho}u_2^2 + 2\rho_2 u_2\tilde{u} + \tilde{p} \\ \tilde{\rho}u_2 v_2 + \rho_2 v_2\tilde{u} + \rho_2 u_2\tilde{v} \\ \tilde{\rho}u_2 w_2 + \rho_2\tilde{u}w_2 + \rho_2 u_2\tilde{w} \\ (e_2 + p_2)\tilde{u} + u_2(\tilde{e} + \tilde{p}) \end{bmatrix}, \quad \tilde{\mathbf{F}} = \begin{bmatrix} \rho_2\tilde{v} + \tilde{\rho}v_2 \\ \tilde{\rho}u_2 v_2 + \rho_2 v_2\tilde{u} + \rho_2 u_2\tilde{v} \\ \tilde{\rho}v_2^2 + 2\rho_2 v_2\tilde{v} + \tilde{p} \\ \tilde{\rho}v_2 w_2 + \rho_2\tilde{v}w_2 + \rho_2 v_2\tilde{w} \\ (e_2 + p_2)\tilde{v} + v_2(\tilde{e} + \tilde{p}) \end{bmatrix} \tag{28}$$

and $[\overline{\mathbf{Q}}] = \overline{Q}_1 - \overline{Q}_2$, $[\overline{\mathbf{E}}] = \overline{E}_1 - \overline{E}_2$, $[\overline{\mathbf{G}}] = \overline{G}_1 - \overline{G}_2$. These equations are in dimensional form, which is valid for both calorically perfect gas and LTE flows.
It was found that the shock has a stabilizing influence only below a certain wave number threshold and no substantial effect above it. Herbert and Esfahanian [55] conjectured that the effect of shock is evident only for wavelengths equal to or greater than the shock height. The wavenumber threshold was estimated to be

$$\alpha_{tr} \approx 2\pi \frac{cos\phi}{y_s} \tag{29}$$

The stabilizing behavior was further confirmed by Pinna and Rambaud [57] also for reacting flows in chemical equilibrium where the final growth rates are lower than the ones obtained for a calorically perfect gas (see Fig. 6). The approximated relation of eq.(29) holds also in those cases. Depending on the specific conditions, this effect may be of interest in the determination of the N-factor (for a detailed review of the e^N method see Arnal [58]) although this is seldom taken into account.

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A LINEAR STABILITY EQUATION FOR COMPRESSIBLE FLOWS

In case one wants to analyze a profile fulfilling the boundary layer assumption, the relation $\overline{P}\gamma M^2$ holds and pressure does not change in the normal-to-the-wall direction, resulting in $d\overline{P}/dy = 0$. By substituting them in the present equations, a standard boundary layer with constant speed at its outer edge can be analyzed.

Continuity equation

$$\begin{aligned}&\frac{i\,\beta\,\gamma\,M^2\,\tilde{p}\,\overline{W}}{\overline{T}} - \frac{i\,\beta\,\gamma\,\tilde{T}\,M^2\,\overline{P}\,\overline{W}}{\overline{T}^2} + \frac{i\,\alpha\,\gamma\,M^2\,\tilde{p}\,\overline{U}}{\overline{T}} - \frac{i\,\alpha\,\gamma\,\tilde{T}\,M^2\,\overline{P}\,\overline{U}}{\overline{T}^2}\\ &-\frac{\gamma\,\tilde{v}\,M^2\,\overline{P}\,\overline{T}_y}{\overline{T}^2} - \frac{i\,\gamma\,\omega\,M^2\,\tilde{p}}{\overline{T}} + \frac{\gamma\,\tilde{v}\,M^2\,\overline{P}_y}{\overline{T}} + \frac{i\,\beta\,\gamma\,\tilde{w}\,M^2\,\overline{P}}{\overline{T}}\\ &+\frac{\gamma\,\tilde{v}_y\,M^2\,\overline{P}}{\overline{T}} + \frac{i\,\alpha\,\gamma\,\tilde{u}\,M^2\,\overline{P}}{\overline{T}} + \frac{i\,\gamma\,\omega\,\tilde{T}\,M^2\,\overline{P}}{\overline{T}^2} = 0\end{aligned} \tag{30a}$$

x-momentum equation

$$\begin{aligned}&\frac{i\,\beta\,\gamma\,\tilde{u}\,M^2\,\overline{P}\,\overline{W}}{\overline{T}} + \frac{\gamma\,\tilde{v}\,M^2\,\overline{P}\,\overline{U}_y}{\overline{T}} + \frac{i\,\alpha\,\gamma\,\tilde{u}\,M^2\,\overline{P}\,\overline{U}}{\overline{T}} - \frac{i\,\gamma\,\omega\,\tilde{u}\,M^2\,\overline{P}}{\overline{T}} = \frac{\tilde{T}\,\overline{\mu}_{\overline{T}}\,\overline{U}_{y\,y}}{Re}\\ &+\frac{\tilde{T}\,\overline{\mu}_{\overline{T}\,\overline{T}}\,\overline{T}_y\,\overline{U}_y}{Re} + \frac{\tilde{T}_y\,\overline{\mu}_{\overline{T}}\,\overline{U}_y}{Re} + \frac{i\,\alpha\,\tilde{v}\,\overline{\mu}_{\overline{T}}\,\overline{T}_y}{Re} + \frac{\tilde{u}_y\,\overline{\mu}_{\overline{T}}\,\overline{T}_y}{Re} - i\,\alpha\,\tilde{p} - \frac{\alpha\,\beta\,\tilde{w}\,\overline{\mu}}{Re} + \frac{i\,\alpha\,\tilde{v}_y\,\overline{\mu}}{Re}\\ &+\frac{\tilde{u}_{y\,y}\,\overline{\mu}}{Re} - \frac{\beta^2\,\tilde{u}\,\overline{\mu}}{Re} - \frac{2\,\alpha^2\,\tilde{u}\,\overline{\mu}}{Re} - \frac{\alpha\,\beta\,\tilde{w}\,\overline{\lambda}}{Re} + \frac{i\,\alpha\,\tilde{v}_y\,\overline{\lambda}}{Re} - \frac{\alpha^2\,\tilde{u}\,\overline{\lambda}}{Re}\end{aligned} \tag{30b}$$

y-momentum equation

$$\begin{aligned}&\frac{i\,\beta\,\gamma\,\tilde{v}\,M^2\,\overline{P}\,\overline{W}}{\overline{T}} + \frac{i\,\alpha\,\gamma\,\tilde{v}\,M^2\,\overline{P}\,\overline{U}}{\overline{T}} - \frac{i\,\gamma\,\omega\,\tilde{v}\,M^2\,\overline{P}}{\overline{T}} = \frac{i\,\beta\,\tilde{T}\,\overline{\mu}_{\overline{T}}\,\overline{W}_y}{Re} + \frac{i\,\alpha\,\tilde{T}\,\overline{\mu}_{\overline{T}}\,\overline{U}_y}{Re}\\ &+\frac{2\,\tilde{v}_y\,\overline{\mu}_{\overline{T}}\,\overline{T}_y}{Re} + \frac{i\,\beta\,\tilde{w}\,\overline{\lambda}_{\overline{T}}\,\overline{T}_y}{Re} + \frac{\tilde{v}_y\,\overline{\lambda}_{\overline{T}}\,\overline{T}_y}{Re} + \frac{i\,\alpha\,\tilde{u}\,\overline{\lambda}_{\overline{T}}\,\overline{T}_y}{Re} - \tilde{p}_y + \frac{i\,\beta\,\tilde{w}_y\,\overline{\mu}}{Re} + \frac{2\,\tilde{v}_{y\,y}\,\overline{\mu}}{Re}\\ &-\frac{\beta^2\,\tilde{v}\,\overline{\mu}}{Re} - \frac{\alpha^2\,\tilde{v}\,\overline{\mu}}{Re} + \frac{i\,\alpha\,\tilde{u}_y\,\overline{\mu}}{Re} + \frac{i\,\beta\,\tilde{w}_y\,\overline{\lambda}}{Re} + \frac{\tilde{v}_{y\,y}\,\overline{\lambda}}{Re} + \frac{i\,\alpha\,\tilde{u}_y\,\overline{\lambda}}{Re}\end{aligned} \tag{30c}$$

z-momentum equation

$$\begin{aligned}&\frac{\gamma\,\tilde{v}\,M^2\,\overline{P}\,\overline{W}_y}{\overline{T}} + \frac{i\,\beta\,\gamma\,\tilde{w}\,M^2\,\overline{P}\,\overline{W}}{\overline{T}} + \frac{i\,\alpha\,\gamma\,\tilde{w}\,M^2\,\overline{P}\,\overline{U}}{\overline{T}} - \frac{i\,\gamma\,\omega\,\tilde{w}\,M^2\,\overline{P}}{\overline{T}} = \frac{\tilde{T}\,\overline{\mu}_{\overline{T}}\,\overline{W}_{y\,y}}{Re} +\\ &+\frac{\tilde{T}\,\overline{\mu}_{\overline{T}\,\overline{T}}\,\overline{T}_y\,\overline{W}_y}{Re} + \frac{\tilde{T}_y\,\overline{\mu}_{\overline{T}}\,\overline{W}_y}{Re} + \frac{\tilde{w}_y\,\overline{\mu}_{\overline{T}}\,\overline{T}_y}{Re} + \frac{i\,\beta\,\tilde{v}\,\overline{\mu}_{\overline{T}}\,\overline{T}_y}{Re} - i\,\beta\,\tilde{p} + \frac{\tilde{w}_{y\,y}\,\overline{\mu}}{Re} - \frac{2\,\beta^2\,\tilde{w}\,\overline{\mu}}{Re}\\ &-\frac{\alpha^2\,\tilde{w}\,\overline{\mu}}{Re} + \frac{i\,\beta\,\tilde{v}_y\,\overline{\mu}}{Re} - \frac{\alpha\,\beta\,\tilde{u}\,\overline{\mu}}{Re} - \frac{\beta^2\,\tilde{w}\,\overline{\lambda}}{Re} + \frac{i\,\beta\,\tilde{v}_y\,\overline{\lambda}}{Re} - \frac{\alpha\,\beta\,\tilde{u}\,\overline{\lambda}}{Re}\end{aligned} \tag{30d}$$

energy equation

$$\frac{i\,\beta\,\gamma\,\tilde{T}\,M^2\,\overline{P}\,\overline{W}}{\overline{T}} + \frac{i\,\alpha\,\gamma\,\tilde{T}\,M^2\,\overline{P}\,\overline{U}}{\overline{T}} + \frac{\gamma\,\tilde{v}\,M^2\,\overline{P}\,\overline{T}_y}{\overline{T}} - \frac{i\,\gamma\,\omega\,\tilde{T}\,M^2\,\overline{P}}{\overline{T}} = \frac{(\gamma-1)\,\tilde{T}\,M^2\,\overline{\mu}_{\overline{T}}\,\left(\overline{W}_y\right)^2}{Re}$$
$$+\frac{2\,(\gamma-1)\,\tilde{w}_y\,M^2\,\overline{\mu}\,\overline{W}_y}{Re} + \frac{2\,i\,\beta\,(\gamma-1)\,\tilde{v}\,M^2\,\overline{\mu}\,\overline{W}_y}{Re} + i\,\beta\,(\gamma-1)\,M^2\,\tilde{p}\,\overline{W} + \frac{(\gamma-1)\,\tilde{T}\,M^2\,\overline{\mu}_{\overline{T}}\,\left(\overline{U}_y\right)^2}{Re}$$
$$+\frac{2\,i\,\alpha\,(\gamma-1)\,\tilde{v}\,M^2\,\overline{\mu}\,\overline{U}_y}{Re} + \frac{2\,(\gamma-1)\,\tilde{u}_y\,M^2\,\overline{\mu}\,\overline{U}_y}{Re} + i\,\alpha\,(\gamma-1)\,M^2\,\tilde{p}\,\overline{U} + \frac{\tilde{T}\,\overline{k}_{\overline{T}}\,\overline{T}_{y\,y}}{Pr\,Re}$$
$$+\frac{\tilde{T}\,\overline{k}_{\overline{T}\,\overline{T}}\,\left(\overline{T}_y\right)^2}{Pr\,Re} + \frac{2\,\tilde{T}_y\,\overline{k}_{\overline{T}}\,\overline{T}_y}{Pr\,Re} + \frac{\tilde{T}_{y\,y}\,\overline{k}}{Pr\,Re} - \frac{\beta^2\,\tilde{T}\,\overline{k}}{Pr\,Re} - \frac{\alpha^2\,\tilde{T}\,\overline{k}}{Pr\,Re}$$
$$-i\,(\gamma-1)\,\omega\,M^2\,\tilde{p} + (\gamma-1)\,\tilde{v}\,M^2\,\overline{P}_y \tag{30e}$$

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